

# Hyperbolicity and Quasi-hyperbolicity in Polynomial Diffeomorphisms of $\mathbb{C}^2$

Eric Bedford and John Smillie

**§0. Introduction.** The iteration of holomorphic mappings may be approached from the dynamical and the analytic/geometric points of view. One of the challenges of holomorphic dynamics is to clarify the interplay between these points of view. Analytically we can look, for instance, where the iterates  $\{f^n : n \geq 0\}$  are (or are not) locally equicontinuous. Dynamically, we can look for expansion/contraction along orbits. By Friedland-Milnor [FM] we know that the dynamically interesting polynomial diffeomorphisms of  $\mathbb{C}^2$  are given by the finite compositions of complex Hénon mappings.

We start with the basic dichotomy into bounded/unbounded orbits;  $K^\pm$  are the sets where the forward/backward orbits are bounded. Thus  $K^\pm$  and  $K := K^+ \cap K^-$  are the analogues of the filled Julia sets for polynomial maps of  $\mathbb{C}$ . The sets where the forward/backward iterates are not equicontinuous are given by  $J^\pm := \partial K^\pm$ . These are the analogues of the Julia set in  $\mathbb{C}$  and are the complements of the sets of Lyapunov stability in forward/backward time. The chaotic dynamics takes place inside the set  $J := J^+ \cap J^-$ .

Another analogue of the Julia set in dimension two is the boundary  $J^* := \partial_S K \subset J$ , where  $\partial_S$  denotes the Shilov boundary (in the sense of function algebras). In [BS3] we showed that  $J^*$  is equal to the closure of the set  $\mathcal{S}$  of saddle periodic points. Additional dynamical properties/characterizations for  $J^*$  were obtained in [BS1,3] and [BLS].

There are several reasons to be interested in hyperbolic polynomial diffeomorphisms. Our understanding of chaotic dynamical systems is most complete in the hyperbolic case. It is also interesting to know how the locus of hyperbolic maps sits in the parameter space and the relation between hyperbolicity and structural stability (see [DL] and [BD]). We take hyperbolicity to mean that the system is hyperbolic on its chain recurrent set. For polynomial diffeomorphisms this is equivalent to the condition that the set  $J$  is a hyperbolic set, though the chain recurrent set may be larger than  $J$ , (see [BS1]). Further, it was shown that  $J = J^*$  for hyperbolic maps, and the stable manifolds  $\mathcal{W}^s$  give a Riemann surface lamination of  $J^+$ . In fact, [BS8, Theorem 8.3] characterized hyperbolicity on  $J$  in terms of the existence of transverse Riemann surface laminations of  $J^+$  and  $J^-$  in a neighborhood of  $J$ . We will strengthen this characterization in Theorem 4.4.

Hyperbolicity involves uniform expansion and contraction, as well as transversality between expanding and contracting directions. In [BS8] we defined a canonical metric  $\|\cdot\|_q^\#$  on the unstable space  $E_q^u \subset T_q \mathbb{C}^2$  for each saddle  $q \in \mathcal{S}$ . A map is said to be *quasi expanding* if  $Df$  expands this metric uniformly (independently of  $q \in \mathcal{S}$ ) in the strong sense that there is a  $\kappa > 1$  with  $\|Df_q v\|_{f(q)}^\# \geq \kappa \|v\|_q^\#$  for any nonzero  $v \in E_q^u$ . For quasi expanding maps, this extends to  $x \in J^*$ . In [BS8] it was shown that every hyperbolic map is quasi expanding, with  $\|\cdot\|^\#$  equivalent to the Euclidean norm. A map is *quasi contracting* if its inverse is quasi expanding.

We let  $B(q, r)$  denote the ball of radius  $r$  centered at  $q$  and let  $W_{q,r}^s$  denote the connected component of  $W_q^s \cap B(q, r)$  containing  $x$ . A geometric characterization of quasi expansion is the Proper, Bounded Area Condition: there exists  $r > 0$  such that for all saddles  $q \in \mathcal{S}$ , (i)  $W_{q,r}^u$  is proper, i.e., closed in  $B(q, r)$ , and (ii) the area of  $W_{q,r}^u$  is uniformly bounded. This means that the degree of local folding of the manifolds  $\{W_q^u : q \in \mathcal{S}\}$  near a saddle point  $q_0$  will be

bounded. On the other hand, if  $\{W_q^u : q \in \mathcal{S}\}$  is part of a lamination, then there is no local folding.

Quasi expansion may be viewed as a 2-dimensional analogue of semi-hyperbolicity for polynomial maps of  $\mathbb{C}$  (see the Appendix of [BS8]). An important motivation for us was the work of Carleson, Jones and Yoccoz [CJY], who showed that semi-hyperbolicity was equivalent to a number of geometric and potential-theoretic properties.

A map will be said to be *quasi-hyperbolic* if it is both quasi expanding and quasi contracting (with no requirement of transversality between stable and unstable directions). Quasi hyperbolic maps share many properties with hyperbolic ones, and we use this to apply hyperbolic methods in more general contexts. In the non-hyperbolic case, the canonical metric  $\|\cdot\|^\#$  may not be equivalent to the euclidean metric on  $\mathbb{C}^2$ . However there is a useful filtration of  $J^*$  by a finite number of sets  $J_{m^s, m^u}^*$ , each of which carries a metric  $\|\cdot\|_{\#}^{m^s, m^u}$ . This metric  $\|\cdot\|_{\#}^{m^s, m^u}$  is equivalent to Euclidean metric for  $x \in J_{m^s, m^u}^*$ , but it blows up as  $x$  approaches a stratum with larger values of  $(m^s, m^u)$ . For maximal values of  $(m^s, m^u)$ ,  $J_{m^s, m^u}^*$  is compact and a uniformly hyperbolic set.

In Theorem 1 we represent stable manifolds as holomorphic mappings  $\xi_x^s : \mathbb{C} \rightarrow \mathbb{C}^2$ , and we write  $W_x^s := \xi_x^s(\mathbb{C})$ , with  $\mathcal{W}^s := \{W_x^s : x \in J^*\}$ . By part (ii) of Theorem 1,  $\{W_q^s : q \in \mathcal{S}\}$  extends continuously to  $\mathcal{W}^s$ . Each  $W_x^s$  is contained in the stable set

$$\mathbb{W}_x^s := \{z \in \mathbb{C}^2 : \lim_{n \rightarrow \infty} \text{dist}(f^n(x), f^n(z)) = 0\}$$

which, dynamically, is the attracting basin of  $x$ . Because of Theorem 3.3 and Corollary 3.4, we call  $W_x^s$  a stable manifold (as is done in [BD]) even though we do not know yet whether  $W_x^s$  always coincides with  $\mathbb{W}_x^s$ .

**Theorem 1.** *Suppose that  $f$  is quasi-contracting. With the notation above, we have*

- (i) *For each  $x \in J^*$ , there is an injective holomorphic immersion  $\xi_x^s : \mathbb{C} \rightarrow J^+ \subset \mathbb{C}^2$  such that  $W_x^s \subset \mathbb{W}_x^s$ .*
- (ii) *There exists  $r_0 > 0$  such that if  $0 < r < r_0$ , then  $W_{x,r}^s$  is a (closed) subvariety of  $B(x, r)$ , and the dependence of the closures  $J^* \ni x \mapsto \overline{W_{x,r}^s}$  is continuous in the Hausdorff topology.*

If  $f$  is quasi-expanding, then Theorem 1 holds for  $\xi_x^u$  and  $W_x^u$ . We write  $\mathcal{W}^{s/u}$  for the families of the stable/unstable manifolds. These stable/unstable manifolds are smooth so it makes sense to say that they have transverse or tangential intersection. With this, we may characterize uniform hyperbolicity.

**Theorem 2.** *Suppose that  $f$  is quasi-hyperbolic. Then  $f$  is uniformly hyperbolic on  $J^*$  if and only if there is no tangency between  $\mathcal{W}^s$  and  $\mathcal{W}^u$ .*

Theorem 2 was proved earlier in [BSr] for real Hénon maps of maximal entropy. This condition is equivalent to  $J \subset \mathbb{R}^2$ , and by [BS8] such maps are quasi hyperbolic. One purpose of the present paper is to extend the work of [BSr] from the context of real, maximal entropy to the more general setting of quasi hyperbolicity.

Theorem 1 will be proved in §3, and Theorem 2 will be proved in §4.

**§1. Invariant families of parametrized curves.** [BS8] gives several distinct but equivalent ways of defining quasi-expansion. One of these is the Proper, Bounded Area Condition, which makes no explicit reference to expansion. There are also a number of other definitions which use the pluri-complex Green function  $G^+$ , which is characterized by the properties:

- (i)  $G^+$  is continuous on  $\mathbb{C}^2$ ,  $G^+ = 0$  on  $K^+$ , and  $G^+ > 0$  on  $\mathbb{C}^2 - K^+$ ,
- (ii)  $G^+$  is pluri-subharmonic on  $\mathbb{C}^2$ , and pluri-harmonic on  $\mathbb{C}^2 - K^+$ ,
- (iii)  $G^+(z) \leq \log(\|z\| + 1) + O(1)$ , and  $\limsup_{z \rightarrow \infty} G^+(z)/\log(\|z\| + 1) = 1$ .

A convenient formula (see [H], [FS], [BS1]) is that  $G^+$  is the super-exponential rate of escape of orbits to infinity:  $G^+(z) = \lim_{n \rightarrow \infty} (\deg(f))^{-n} \log(\|f^n(z)\| + 1)$ .

One of the equivalent definitions of quasi-expansion concerns the existence of a large normal family of entire curves imbedded in  $J^-$ . We will use this as our definition and derive a number of its properties. We let  $\Psi$  be an  $f$ -invariant family of injective holomorphic maps  $\psi : \mathbb{C} \rightarrow J^- \subset \mathbb{C}^2$  such that  $\psi'(\zeta) \neq 0$  for all  $\zeta \in \mathbb{C}$  and which satisfy the disjointness (1.1) and normalization (1.2) conditions:

$$\text{If } \psi_1, \psi_2 \in \Psi, \text{ then either } \psi_1(\mathbb{C}) = \psi_2(\mathbb{C}), \text{ or } \psi_1(\mathbb{C}) \cap \psi_2(\mathbb{C}) = \emptyset \quad (1.1)$$

and

$$\max_{|\zeta| \leq 1} G^+(\psi(\zeta)) = 1, \quad (1.2)$$

We note that if  $\psi$  satisfies (1.2), then so does the “rotated” parametrization  $\psi(e^{ia}\zeta)$  for any  $a \in \mathbb{R}$ . If  $x = \psi(0)$ , then by the disjointness condition, we may write  $\psi = \psi_x$ . Any other  $\phi \in \Psi$  with  $\phi(0) = x$  and which also satisfies (1.2) will be rotated reparametrization of  $\psi_x$ . Thus the image of  $\psi_x$  determines the map  $\psi_x$  up to precomposition by a rotation of  $\mathbb{C}$ .

Let  $X \subset J$  be a set with  $J^* \subset \overline{X}$ . We let  $\Psi := \{\psi_x : x \in X\}$  be an invariant family as above. We give the two principal examples of families  $\Psi$ , which exist for all Hénon maps. For one of them,  $\Psi_{\mathcal{S}}$ , the curves are the unstable manifolds of saddle points and are thus pairwise disjoint. This is Example 1 below, and is what was used in [BS8]. For Example 2 we work inside a single slice: all of the curves are the same unstable manifold  $W^u(q)$ , but the parametrizations are different.

**Example 1. (Saddle Points)** We let  $X := \mathcal{S}$  be the set of periodic points of saddle type. Suppose that  $f^N(p) = p$ , and the multipliers of  $Df^N$  at  $p$  are  $\nu_s, \nu_u \in \mathbb{C}$  with  $|\nu_s| < 1 < |\nu_u|$ . For each  $p \in \mathcal{S}$ , there is a uniformization  $\xi_p : \mathbb{C} \rightarrow W^u(p)$  such that  $\xi_p(0) = p$ , and  $f^N(\xi_p(\zeta)) = \xi_p(\nu_u \zeta)$ . We may change the parametrization so that  $\psi_p(\zeta) := \xi_p(\alpha \zeta)$  satisfies (1.2). It follows that  $\Psi_{\mathcal{S}} := \{\psi_p : p \in \mathcal{S}\}$  satisfies the conditions above.

**Example 2. (Recentered Unstable Manifold)** Let  $q$  denote any saddle point, and let  $\xi_q : \mathbb{C} \rightarrow W^u(q)$  be the uniformization. By [BLS],  $W^u(q) \cap W^s(q)$  is a dense subset of  $J^*$ , and we choose  $X \subset W^u(q) \cap J$  such that  $J^* \subset \overline{X}$ . For  $y \in X$ , let  $\zeta_y \in \mathbb{C}$  be such that  $\xi_q(\zeta_y) = y$ . We may “re-center” the parametrization of this curve to the point  $y$ , i.e., we choose  $\alpha \in \mathbb{C}$  so that  $\psi_y(\zeta) := \xi_q(\alpha \zeta + \zeta_y)$  satisfies (1.2). Thus  $\Psi_{q\mathcal{R}} := \{\psi_y : y \in X\}$  satisfies the conditions above.

We say that  $f$  is *quasi expanding\** if  $\Psi$  is a normal family, which means that if  $\{\psi_j\} \subset \Psi$  is any sequence, then there is a subsequence  $\{\psi_{j_k}\}$  which converges uniformly on compact subsets of  $\mathbb{C}$  to an entire map  $\psi : \mathbb{C} \rightarrow \mathbb{C}^2$ . We let  $\widehat{\Psi}$  denote the set of such normal limits, and for  $x \in \overline{X}$  we set  $\widehat{\Psi}_x := \{\psi \in \widehat{\Psi} : \psi(0) = x\}$ . There are quasi-expanding maps  $f$  for which  $\psi \in \widehat{\Psi}$  may fail to be 1-to-1, and there may be  $\zeta_0$  where  $\psi'(\zeta_0) = 0$ . Thus the elements of  $\widehat{\Psi}_x$  may not be essentially unique modulo rotation of parameters.

---

\* Our definition of quasi expansion is slightly different from [BS8] because of the introduction of the set  $X$ , which allows us to deal more flexibly with the possibility that  $J^* \neq J$ .

**Proposition 1.1.** *If  $f$  is quasi-expanding, then  $\widehat{\Psi}$  satisfies (1.1) and (1.2).*

*Proof.* It is evident that (1.2) must hold. If (1.1) fails, there are  $\zeta_1, \zeta_2 \in \mathbb{C}$  and  $\psi_1, \psi_2 \in \widehat{\Psi}$  such that  $\psi_1(\zeta_1) = \psi_2(\zeta_2)$ , but  $\psi_1(\mathbb{C}) \neq \psi_2(\mathbb{C})$ . Thus  $\tilde{x} := \psi_1(\zeta_1) = \psi_2(\zeta_2)$  is an isolated point of  $\psi_1(\mathbb{C}) \cap \psi_2(\mathbb{C})$ . On the other hand,  $\psi_1$  is the locally uniform limit of  $\psi_{1,j} \in \Psi$  (and similarly for  $\psi_2$ ). By the continuity of complex intersections, there is an intersection point of  $\psi_{1,j}(\mathbb{C}) \cap \psi_2(\mathbb{C})$  near  $\tilde{x}$  when  $j$  is sufficiently large. Now if we choose  $k$  sufficiently large, there is intersection point of  $\psi_{1,j}(\mathbb{C}) \cap \psi_{2,k}(\mathbb{C})$  near  $\tilde{x}$ . This contradicts (1.1) for the original family  $\Psi$ .  $\square$

For  $x \in \overline{X}$  and  $r > 0$  we let  $B(x, r) \subset \mathbb{C}^2$  denote the ball of radius  $r$ , centered at  $x$ . For  $\psi \in \widehat{\Psi}_x$ , we let  $\mathcal{D}_\psi = \mathcal{D}_\psi(r)$  denote the connected component of the open set  $\psi^{-1}(B(x, r))$  containing the origin. By (1.2),  $\psi$  is non-constant, and we can choose  $r > 0$  small enough that  $\psi : \mathcal{D}_\psi \rightarrow B(x, r)$  is proper, which corresponds to the property that  $|\psi| = r > 0$  on  $\partial\mathcal{D}_\psi$ . We may choose a uniform  $r$  that works for all  $x \in J^*$ . By the Maximum Principle,  $\mathcal{D}_\psi$  is a topological disk, so by the Riemann Mapping Theorem it is conformally equivalent to the unit disk. For  $x \in \overline{X}$  and  $\psi \in \widehat{\Psi}_x$ , the fact that  $\psi$  is non constant means that there is an integer  $m \geq 1$  and  $\vec{a} \in \mathbb{C}^2$  such that  $\psi(\zeta) = x + \vec{a}\zeta^m + \dots$ , where the dots indicate higher powers of  $\zeta$ . Thus there are  $r, \rho > 0$  such that  $\text{dist}(\psi(\zeta), x) \geq r$  for all  $|\zeta| = \rho$ . This means that  $\mathcal{D}_\psi \subset \{|\zeta| < \rho\}$ . Since  $|\psi'|$  is bounded by some number  $M$  on  $|\zeta| \leq \rho$ , we know that  $\{|\zeta| < r/M\} \subset \mathcal{D}_\psi$ . By the compactness of  $\widehat{\Psi}$ , we have:

**Proposition 1.2.** *For sufficiently small  $r > 0$ , there are  $0 < \rho_1 < \rho_2$  such that  $\{|\zeta| < \rho_1\} \subset \mathcal{D}_\psi \subset \{|\zeta| < \rho_2\}$  for all  $\psi \in \widehat{\Psi}$ .*

Let  $r$  be as in Proposition 1.2, and let  $x \in \overline{X}$  and  $\psi \in \widehat{\Psi}_x$ . As in §0, we define  $W_{x,r}^u := \psi(\mathcal{D}_\psi)$ , and we denote the family of these disks by  $\mathcal{W}_r^u := \{W_{x,r}^u : x \in \overline{X}\}$ .

**Proposition 1.3.** *The set  $W_x^u$  is well defined and does not depend on the choice of  $\psi \in \widehat{\Psi}_x$ , and  $\psi : \mathcal{D}_\psi \rightarrow B(x, r)$  is a proper imbedding. Further,  $W_x^u$  has the following properties:*

- (i)  $W_{x,r}^u$  is a nonsingular subvariety of  $B(x, r)$ .
- (ii) The family  $\mathcal{W}_r^u$  has uniformly bounded area:  $\sup_{x \in \overline{X}} \text{Area}(W_{x,r}^u) < \infty$ .
- (iii)  $\overline{X} \ni x \mapsto \overline{W_{x,r}^u}$  is continuous in the Hausdorff topology.

*Proof.* We start by proving uniqueness of the set  $W_x^u$ . By the construction of  $\mathcal{D}_\psi$  given above, it follows that  $\psi : \mathcal{D}_\psi \rightarrow B(x, r)$  is proper. If there are  $\psi_1, \psi_2 \in \widehat{\Psi}_x$  with  $W_{\psi_1}^u \neq W_{\psi_2}^u$ , then there are points  $\zeta_1, \zeta_2 \in \mathbb{C}$  with  $0 < |\zeta_1|, |\zeta_2| < \rho_2$  such that  $\psi_1(\zeta_1) = \psi_2(\zeta_2)$ . Now for  $j = 1, 2$ ,  $\psi_j$  is the limit of maps  $\psi_j^{(n)}$ ,  $n \rightarrow \infty$ . As in the proof of Proposition 1.1, there are points  $\zeta_j^{(n)}$  with  $\zeta_j^{(n)} \rightarrow \zeta_j$  as  $n \rightarrow \infty$ , and  $\psi_1^{(n)}(\zeta_1^{(n)}) = \psi_2^{(n)}(\zeta_2^{(n)})$ . This, however, violates (1.1).

The proof of item (i) is nontrivial and is a consequence of Proposition 12 of [LP]. Items (ii) and (iii), on the other hand, are easy consequences of the normality of  $\Psi$ . In particular, for (iii) we use the uniqueness of  $W_{x,r}^u$ , together with the fact that  $\psi_x : \mathcal{D}_\psi \rightarrow B(x, r)$  is proper. Thus convergence of  $\overline{W_{x,r}^u}$  in Hausdorff topology is a consequence of uniform convergence of  $\psi_x$ .  $\square$

**§2. Expanded metric.** We continue to work with  $\Psi$  as above. If  $x \in X$ , then  $f(\psi_x)$  differs from  $\psi_{f(x)}$  by a linear reparametrization, so there exists  $\lambda_x \in \mathbb{C}$  such that

$$f(\psi_x(\zeta)) = \psi_{f(x)}(\lambda_x \zeta). \quad (2.1)$$

For  $\lambda \in \mathbb{C}$ , we define the linear map  $L_\lambda(\zeta) := \lambda\zeta$ . The composition  $\psi_{f(x)}^{-1} \circ f \circ \psi_x : \mathbb{C}_x \rightarrow \mathbb{C}_{f(x)}$  is an invertible holomorphic map fixing the origin, so by (2.1), we have  $\psi_{f(x)}^{-1} \circ f \circ \phi_x = L_{\lambda_x}$  for some  $\lambda_x \in \mathbb{C}$ . This means that the action of  $f$  on the curves  $\psi_x(\mathbb{C})$  is that the curve through  $x$  is taken to the curve through  $f(x)$ , and the uniformizing parameters precompose by the linear maps  $L_{\lambda_x} : \mathbb{C}_x \rightarrow \mathbb{C}_{f(x)}$ .

The condition (1.2) may be interpreted as saying that the unit disk  $\{|\zeta| < 1\}$  is the largest disk centered at the origin and contained in the set  $\{G^+ \circ \psi_x < 1\} \subset \mathbb{C}_x$ . Under  $f$  (equivalently  $L_{\lambda_x}$ ) this is taken to the disk  $\{|\zeta| < |\lambda_x|\} \subset \mathbb{C}_{f(x)}$ . A basic property of  $G^+$  is that  $G^+ \circ f = \deg(f)G^+$ , and  $\deg(f) \geq 2 > 1$ . Since  $\{|\zeta| < |\lambda_x|\}$  is then the largest disk inside  $\{G^+ < \deg(f)\}$ , it follows that  $|\lambda_x| > 1$ .

Now we give a second interpretation of  $|\lambda_x|$ . For each  $x \in X$ , we define  $E_x^u$  to be the 1-dimensional subspace of  $T_x(\mathbb{C}^2)$  spanned by  $\psi'_x(0)$ . We let  $|\cdot|_e$  denote the Euclidean norm and define a new norm on  $E_x^u$ :

$$\|v\|_x^\# := \frac{|v|_e}{|\psi'_x(0)|_e}, \quad v \in E_x^u.$$

With this definition, the (operator) norm of  $\psi'_x(0)$  is 1. By the chain rule, we have

$$D_x f \cdot \psi'_x(0) = \psi'_{f(x)}(0) \cdot \lambda_x \quad (2.2)$$

Thus the operator norm of the restriction of  $D_x f$  to  $E_x^u$  with respect to the metric  $\|\cdot\|_x^\#$  is given by

$$\|D_x f\|_x^\# = \frac{\|D_x f(v)\|_{f(x)}^\#}{\|v\|_x^\#} = |\lambda_x|, \quad \text{for } v \in E_x^u$$

We define three rate of growth functions:

$$m_\psi(r) := \max_{|\zeta| \leq r} G^+(\psi(\zeta)), \quad m(r) := \inf_{\psi \in \Psi} m_\psi(r), \quad M(r) := \sup_{\psi \in \widehat{\Psi}} m_\psi(r)$$

With this notation, we see that for  $x \in X$ ,  $|\lambda_x|$  is defined by the condition  $m_{\psi_{f(x)}}(|\lambda_x|) = d$ .

**Proposition 2.1.** *If  $f$  is quasi-expanding, then the following hold:*

- (i)  $M(r) < \infty$  for all  $r < \infty$ .
- (ii)  $m(r) > 0$  for all  $r > 0$
- (iii) Let  $\kappa$  be such that  $M(\kappa) = \deg(f)$ . Then  $\kappa > 1$ , and  $|\lambda_x| \geq \kappa$  for all  $x \in X$ .

*Proof.* These properties are proved in [BS8] and follow from the normality of the family  $\Psi$ .  $\square$

In fact, each of these conditions is equivalent to quasi expansion (see [BS8]).

**Proposition 2.2.** *If  $f$  is quasi-expanding, then for sufficiently large  $N$ ,  $f^{-N}$  maps  $\mathcal{W}_r^u$  inside itself. That is, for each  $x \in \overline{X}$ ,  $f^{-N}(W_{x,r}^u) \subset W_{f^{-N}(x),r}^u$ . Further, if  $y \in W_{x,r}^u$ , then  $\text{dist}(f^{-n}(x), f^{-n}(y)) \rightarrow 0$  like a constant times  $\kappa^{-n}$  as  $n \rightarrow +\infty$ .*

*Proof.* Let  $\rho_1, \rho_2$  are as in Proposition 1.2, let  $\kappa$  is as in Proposition 2.1, and let  $N$  be large enough that  $\rho_1 \kappa^N > \rho_2$ . Thus

$$f^{-N}W_{x,r}^u \subset \psi_{f^{-N}(x)}(|\zeta| < \kappa^{-N}\rho_2) \subset \psi_{f^{-N}(x)}(\mathcal{D}_{f^{-N}(x)}) = W_{f^{-N}(x),r}^u$$

For the last assertion, by the normality of  $\Psi$  we have  $|\psi'(\zeta)| \leq C$  for all  $\psi \in \Psi$  and  $|\zeta| \leq \rho_2$ . Now for any two points of  $V_x^u$ , we may write them as  $y' = \psi_x(\zeta')$  and  $y'' = \psi_x(\zeta'')$ . The distance between the  $\zeta$ -coordinates of  $f^{-n}(y')$  and  $f^{-n}(y'')$  is  $\kappa^{-n}|\zeta' - \zeta''|$ . Thus we conclude that  $\text{dist}(f^{-n}(y'), f^{-n}(y'')) \leq \rho_2 \kappa^{-n} C$ , which proves the last assertion.  $\square$

**Proposition 2.3.** *If  $f$  is quasi-expanding, then for  $x \in J^*$  and  $x' \in W_{x,r}^u \cap J^*$  with  $x \neq x'$ , there is an  $N > 0$  such that  $f^n(x') \notin W_{f^n(x),r}^u$  for  $n \geq N$ .*

*Proof.* The proof of this is similar to the proof of the last assertion in Proposition 2.2. The point  $x$  is given by  $\psi_x(0)$ , and there is  $\zeta' \in \mathcal{D}_x$  such that  $x' = \psi_x(\zeta')$ . The  $\zeta$ -coordinate of  $f^n(x')$  has modulus bounded below by  $|\zeta'| \kappa^n$ , which is larger than  $\rho_2$  for  $n$  large, and so it does not belong to  $\mathcal{D}_{f^n(x)}$ . This means that  $f^n(x') \notin W_{f^n(x),r}^u$ .  $\square$

**§3. Stable manifolds.** We will prove Theorem 1 in this section. We continue to assume that  $f$  is quasi-hyperbolic. We have defined  $W_{x,r}^u$  and  $\xi_x^u := \xi_x$  for a quasi-expanding map  $f$ , and we will write  $W_x^s$  and  $\xi_x^s$  for the corresponding local stable manifolds and parametrizing maps for  $f^{-1}$ . When  $f$  is hyperbolic,  $\mathcal{W}^s$  is the family of stable manifolds; and when  $f$  is quasi-hyperbolic but not hyperbolic,  $\mathcal{W}^s$  still has many of the properties of a family of stable manifolds for a hyperbolic diffeomorphism.

**Proposition 3.1.** *If  $f$  is quasi-contracting, then for all  $x \in J^*$ ,  $W_x^s = \bigcup_{n \geq 0} f^{-n} W_{f^n(x),r}^s$  is a manifold consisting of stable points, i.e.,  $W_x^s \subset \mathbb{W}_x^s$ .*

*Proof.* We let  $\rho_1$  and  $\rho_2$  be as in Proposition 1.2. Thus  $W_{f^n(x),r}^s \supset \psi_{f^n(x)}^s(|\zeta| < \rho_1)$ . By quasi-contraction, we have that  $f^{-n}(W_{f^n(x),r}^s) \supset \psi_x^s(|\zeta| < \kappa^n \rho_1)$ . Thus  $W_{x,r}^s$  will contain all of  $\psi_x^s(\mathbb{C})$ . For the second statement, Proposition 2.2, applied to  $f^{-1}$ , shows that  $W_{x,r}^s$ , and thus  $W_x^s$  is contained in  $\mathbb{W}_x^s$ .  $\square$

We will show that the local disks  $W_{x,r}^s$  and  $W_{x,r}^u$  have a weak sort of transversality. For a set  $E \subset \mathbb{C}^2$  and  $\epsilon > 0$ , we will let  $E^\epsilon$  denote the  $\epsilon$ -neighborhood of  $E$ , i.e. set of points of distance less than  $\epsilon$  from  $E$ .

**Lemma 3.2.** *Let  $f$  be quasi-hyperbolic, and let  $r > 0$  be the as in §1. Then for  $r_1 < r_2 < r$  there exist  $r_0, \epsilon > 0$  with the following property. Let  $B_0 \subset B_1 \subset B_2 \subset \mathbb{C}^2$  be concentric balls such that  $B_j$  has radius  $r_j$ . Then*

- (i) *For  $x', x'' \in B_0 \cap J^*$ , the intersections  $W_{x',r}^s \cap B_2$  and  $W_{x'',r}^u \cap B_2$  are closed subsets of  $B_2$ ,*
- (ii)  *$W_{x',r}^s \cap W_{x'',r}^u \cap B_1$  is nonempty, and*
- (iii)  *$(W_{x',r}^s)^\epsilon \cap (W_{x'',r}^u)^\epsilon \subset B_1$ .*
- (iv)  *$\text{diam}(\{x', x''\} \cup (W_{x',r}^s \cap W_{x'',r}^u)) \rightarrow 0$  as  $\text{dist}(x', x'') \rightarrow 0$ .*

*Proof.* First, we know that  $J^* \ni x \mapsto W_{x,r}^u$  is continuous in the Hausdorff topology. Since  $r_2 < r$ , we may take  $r_0 > 0$  sufficiently small that  $W_x^u \cap B_{r_2}$  will be closed in  $B_{r_2}$ . Thus  $W_{x,r}^u \cap B_{r_2}$  and  $W_{x,r}^s \cap B_{r_2}$  are subvarieties of  $B_{r_2}$ . Since these varieties do not actually coincide in a neighborhood of  $x$ , we may choose  $r_2$  small enough that  $W_{x,r}^u \cap W_{x,r}^s \cap B_{r_2} = \{x\}$ . The intersection of complex varieties has a continuity property. Specifically, for any  $r_1 > 0$ , we may choose  $r_0 > 0$  sufficiently small that the intersection  $W_{x'}^u \cap W_{x''}^s \cap B_{r_2}$  is contained in  $B_{r_1}$ . Further, we may choose  $\epsilon > 0$  sufficiently small that the  $\epsilon$ -neighborhood is also contained in  $B_{r_1}$ . Finally, we note that (iii) implies that the intersection points of  $W_{x',r}^s \cap W_{x'',r}^u$  cannot escape to the boundary of  $B_1$ , so (iv) is a consequence of continuity in the Hausdorff topology.  $\square$

If we set  $g := f|_{J^*}$ , then we see that the manifolds  $\mathcal{W}^s$  give the stable sets for  $g$  in the following sense:

**Theorem 3.3.** *If  $f$  is quasi-hyperbolic, then for all  $x \in J^*$ ,  $W_x^s \cap J^* = \mathbb{W}_x^s \cap J^*$ .*

*Proof.* By Proposition 3.1, we need to show the containment ‘ $\supset$ ’. Suppose that  $z \in \mathbb{W}_x^s \cap J^*$ . We will show that for  $n$  sufficiently large, the local varieties  $W_{f^n(z),r}^s$  and  $W_{f^n(x),r}^s$  intersect. Thus by Proposition 3.1, we have  $z \in W_x^s$ .

Now let  $r_0, r_1, r_2$  and  $\epsilon$  be as in Lemma 3.2. There exists  $N$  sufficiently large that  $f^n(z) \in B(f^n(x), r_0)$  for all  $n \geq N$ . We may suppose that  $W_{f^n(x),r}^s$  and  $W_{f^n(z),r}^s$  are disjoint, for otherwise we would have  $W_{f^n(x)}^s = W_{f^n(z)}^s$ , and thus  $W_x^s = W_z^s$ . By Lemma 3.2, the set  $S_n := W_{f^n(x),r}^u \cap W^s(f^n(z), r)$  is nonempty for all  $n \geq N$ . On the other hand,  $f^j(S_n) = S_{n+j}$ , which contradicts Proposition 2.3.  $\square$

We also have a local uniqueness result for local varieties.

**Corollary 3.4.** *Let  $f$  be quasi-hyperbolic, and let  $x \in J^*$ . If  $U$  is a sufficiently small neighborhood of  $x$ , and  $V_x$  is an irreducible subvariety of  $U$  with  $V_x \cap U \subset \mathbb{W}_x^s$ , then  $W_{x,r}^s \cap U = V_x \cap U$ .*

*Proof.* By Theorem 3.3, we know that  $J^* \cap V_x = J^* \cap W_{x,r}^s$ . Since  $J^*$  is an infinite set with accumulation points, the result follows.  $\square$

We conclude this section with a result that is proved much along the lines of Theorem 9.6 of [BLS]:

**Theorem 3.5.** *Let  $f$  be quasi-hyperbolic, and let  $T_1, T_2 \subset J^*$  be closed, invariant subsets. Then  $\bigcup_{x \in T_1} W_x^s \cap J^*$  and  $(\bigcup_{x \in T_1} W_x^s) \cap (\bigcup_{y \in T_2} W_y^u)$  are dense subsets of  $J^*$ .*

**§4. Characterization of hyperbolicity.** This section is devoted to a proof of Theorem 2.\* We start by recalling some notation and results from [BS8]. For  $\psi : \mathbb{C} \rightarrow \mathbb{C}^2$ , we define the order at  $\zeta = 0$  to be  $\text{Ord}(\psi) := \min\{m \geq 1 : \psi^{(m)}(0) \neq 0\}$ . For  $x \in J^*$ , we define  $\tau^{s/u}(x) = \max_{\psi \in \hat{\Psi}_x^{s/u}} \text{Ord}(\psi)$ . The quantity  $\tau^u(x)$  measures the failure of  $\mathcal{W}^u$  to be a lamination in a neighborhood. More precisely, let  $\pi : \mathbb{C}^2 \rightarrow E_x^u$  denote a linear projection. Then for  $y \in J^*$  close to  $x$ ,  $\pi : W_{y,r}^u \rightarrow E_x^u$  is a branched cover over a neighborhood of  $x$ . By *local folding*, we mean that there will be  $y$  near  $x$  for which the local branching order is  $\tau^u(x)$ . Thus there is no neighborhood of  $x$  on which  $\mathcal{W}^u$  can be a lamination. Looking at  $\mathcal{W}^s$  at the same time, we see that there will be  $y', y'' \in J^*$  such that  $W_{y',r}^s \cap W_{y'',r}^u$  contains  $\tau^s(y')\tau^u(y'')$  points near  $x$ .

Consider  $\psi_x \in \Psi_x^u$  with  $j = \text{Ord}(\psi_x)$ . We have  $\psi_x(\zeta) = x + a_j \zeta^j + \dots$ , where  $a_j \in \mathbb{C}^2$  is nonzero, and ‘ $\dots$ ’ indicates terms of higher order. Further,  $a_j = \psi^{(j)}(0)/j!$  spans the unstable tangent space  $E_x^u$ . Let us define

$$\gamma^u(x) := \inf_{\psi_x \in \hat{\Psi}_x^u, \text{Ord}(\psi_x) = \tau^u(x) = j} \left| \frac{\psi^{(j)}(0)}{j!} \right|_e$$

where  $|\cdot|_e$  denotes the euclidean metric on  $\mathbb{C}^2$ . For  $x \in J^*$  with  $\tau^u(x) = j$ , we define a metric on a vector  $v \in E_x^u$  by

$$\|v\|_x^{\#,j} := \frac{|v|_e}{\gamma^u(x)}$$

---

\* It might appear that we could obtain a result stronger than is stated in Theorem 2, since the paper [F] claims that: If  $J^*$  is a hyperbolic set, then  $J = J^*$ . However, the proof given in [F] seems to be incomplete.

This metric is uniformly expanded on the set  $\{x \in J^* : \tau^u(x) = j\}$ , but it need not be bounded with respect to  $|\cdot|_e$  as  $x$  approaches points  $x' \in J^*$  where  $\tau^u(x') > j$ . We define

$$J_{m^s, m^u}^* := \{x \in J^* : \tau^s(x) = m^s, \tau^u(x) = m^u\}.$$

We say that the pair  $(m^s, m^u)$  is *maximal* if  $J_{m^s, m^u}^* \neq \emptyset$ , and whenever  $m_1^s \geq m^s$  and  $m_1^u \geq m^u$ , then  $J_{m_1^s, m_1^u}^* = \emptyset$  unless  $m_1^s = m^s$  and  $m_1^u = m^u$ . If  $(m^s, m^u)$  is maximal, the metrics  $\|\cdot\|^{\#, m^s}$ , and  $\|\cdot\|^{\#, m^u}$  are equivalent to  $|\cdot|_e$  on  $E_x^s$  and  $E_x^u$  on  $J_{m^s, m^u}^*$ . It follows that:  $J_{m^s, m^u}^*$  is a compact, hyperbolic set whenever  $(m^s, m^u)$  is a maximal pair. From [BS8] we have:

**Theorem 4.1.**  $J_{1,1}^*$  is a dense, open subset of  $J^*$ . Further,  $f$  is uniformly hyperbolic on  $J^*$  if and only if  $J_{1,1}^* = J^*$ .

*Proof of Theorem 2.* If  $J_{1,1}^* = J^*$ , then  $f$  is uniformly hyperbolic, and thus the stable and unstable families  $\mathcal{W}^s$  and  $\mathcal{W}^u$  are transverse. Conversely, suppose that  $J_{1,1}^*$  is a proper subset of  $J^*$ . In this case, there is a maximal pair  $(m^s, m^u) \neq (1, 1)$ , and  $J_0^* := J_{m^s, m^u}^*$  is a hyperbolic subset of  $J^*$ . We may suppose that  $m^s > 1$ . Given  $\delta, \rho > 0$ , we cover  $J_0^*$  with finitely many compact sets  $X_\iota$  such that for each  $\iota$  there is a translation and rotation of coordinates so that  $X_\iota \subset \Delta_\delta \times \Delta_\delta$ . We set  $\mathcal{N}_\iota = \Delta_\rho \times \Delta_\rho$ . For  $x \in X_\iota$ , we let  $W_{x, \rho}^{s/u}$  denote the connected component of  $W_{x, \rho}^{s/u} \cap \mathcal{N}_\iota$  containing  $x$ . Shrinking  $\delta, \rho$  if necessary, we may assume that for  $x \in X_\iota$ , each  $W_{x, \rho}^u$  is “horizontal” in  $\mathcal{N}_\iota$ , i.e., a graph over the first coordinate, and each  $W_{x, \rho}^s$  is “vertical”. Set  $\mathcal{W}_{X_\iota}^{s, u} = \{W_{x, r}^{s, u} : x \in X_\iota\}$ . By the hyperbolicity of  $J_0^*$ , there are cone fields with respect to which  $f$  is uniformly expanding in the horizontal direction of  $\mathcal{N}_\iota$  and uniformly contracting in the vertical direction.

Now let us choose a neighborhood  $\mathcal{N}_0$  of  $J_0^*$  such that  $f(\mathcal{N}_0) \subset \bigcup_\iota \mathcal{N}_\iota$ , and  $\tau^s < m^s$  on  $\mathcal{W}_{J_0^*}^u - \mathcal{N}_0$ . This is possible because  $J^* \cap \mathcal{W}_{J_0^*}^u$  is dense in  $J^*$  by Corollary 3.5. If  $\tau^s = m^s$  on  $J^* \cap \mathcal{W}_{J_0^*}^u$ , then by the upper-semicontinuity of  $\tau^s$ , it would follow that  $\tau^s = m^s$  on all of  $J^*$ , which is not possible by Theorem 4.1.

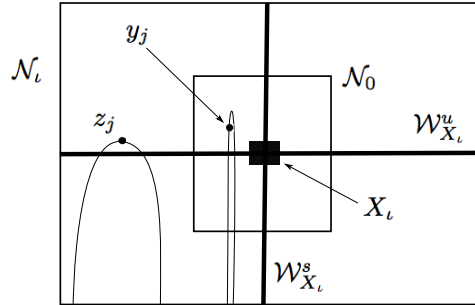


Figure 1.

For  $y_\infty \in J_0^*$  there is a sequence  $(y_j) \subset J^*$  such that  $y_j \rightarrow y_\infty$ , and  $\psi_{y_j}^s \in \Psi_{y_j}^s$  converges to a map  $\psi_\infty^s$  with the property that  $\text{Ord}(\psi_\infty^s) = m^s$ . We may suppose that  $y_\infty \in X_\iota$ , so that  $W_{y_\infty, \rho}^s$  is horizontal in  $\mathcal{N}_\iota$ . In order for  $\psi_\infty^s$  to have order  $m^s > 1$ , it is necessary that the nearby  $W_{y_j, \rho}^s$  are  $m^s$ -fold branched covers with respect to the coordinate projection to the tangent space of  $W_{y_\infty, \rho}^s$ . Since  $\mathcal{W}_{X_\iota}^s$  is a lamination, we see that we cannot have  $y_j \in W_{X_\iota}^s$ . Since  $y_\infty \in J_0^*$ ,  $W_{y_\infty}^u$  is horizontal in  $\mathcal{N}_\iota = \Delta_\rho \times \Delta_\rho$  in the sense that it is a graph over the first factor  $\Delta_\rho$  and bounded away from  $\Delta_\rho \times \partial\Delta_\rho$ . It follows that the intersection  $W_{y_j, r}^s \cap W_{y_\infty, r}^u$



consists of  $m^s$  points. By Proposition 2.2, these points get closer together as we map forward under  $f$ .

Now since  $y_j \notin W_{X_i}^s$ , it follows that  $f^n(y_j)$  cannot remain in  $\mathcal{N}_i \cap \mathcal{N}_0$  for all  $n > 0$ . Thus there is an  $n_j > 0$  such that  $f^k(y_j) \in \mathcal{N}_0$  for  $0 \leq k < n_j$ , and  $f^{n_j}(y_j) \in \mathcal{N}_i - \mathcal{N}_0$ . We may pass to a subsequence so that the points  $z_j := f^{n_j}(y_j)$  all belong to the same set  $\mathcal{N}_i - \mathcal{N}_0$ . By hyperbolicity,  $W_{X_i}^u$  is a family of graphs horizontal in  $\mathcal{N}_i$ , and we conclude that a subsequence of the points  $z_j$  converges to a point  $z_\infty \in W_{X_i}^u$ .

Now we consider the multiplicity of the intersection between  $W_{z_\infty, \rho}^s$  and  $W_{z_\infty, \rho}^u$  at  $z_\infty$ . The set  $W_{y_j, r}^s \cap W_{y_\infty, r}^u$  consists of  $m^s$  points, and by part (iv) of Lemma 3.2, these points coalesce as  $j \rightarrow \infty$ . Thus by the continuity of the intersection multiplicity in the complex setting, the intersection multiplicity is  $m^s$ . On the other hand, let  $\hat{\psi}_\infty^s$  be the limit of the maps  $\psi_{z_j}^s$ . It follows that  $\text{Ord}(\hat{\psi}_\infty^s) \leq \tau^s(z_\infty) < m^s$ . We conclude that the curves must intersect tangentially.  $\square$

**Proposition 4.2.** *If a point  $x \in J_{m^s, m^u}^*$  and  $W^u(x)$  and  $W^s(x)$  are tangent to order  $k$  then the forward limit set of  $x$  is contained in  $J_{p, q}^*$  with  $p \geq m^s$  and  $q \geq (k+1)m^u$  the backward limit set of  $x$  is contained in  $J_{p', q'}^*$  with  $p' \geq (k+1)m^s$  and  $q' \geq m^u$ .*

*Proof.* We prove the first statement. The second statement follows by considering  $f^{-1}$ . Consider the case when  $m^s = m^u = 1$ . Say that  $\psi_x(\zeta)$  parametrizes  $W^u(x)$  and  $\phi_x(\zeta)$  parametrizes  $W^s(x)$ . Since  $W^u(x)$  and  $W^s(x)$  are tangent to order  $k$  we can assume that for some  $c \neq 0$ ,  $\psi_x(\zeta) = \phi_x(c\zeta) + \dots$  where the dots represent terms of order greater than  $k$ . Write  $f^n(\psi_x(\zeta)) = \psi_{f^n(x)}(\lambda_n \zeta)$  and  $f^n(\phi_x(\zeta)) = \phi_{f^n(x)}(\mu_n \zeta)$ . For some  $\kappa > 1$  we have  $\lambda_n \geq \kappa^n$  and  $\mu_n \leq \kappa^{-n}$ .

Now  $f^n(\psi_x(\zeta)) = f^n(\phi_x(c\zeta))$  up to terms of order greater than  $k$ . Write  $\psi_{f^n(x)}(\zeta) = f^n(x) + \vec{a}_{1,n}\zeta + \vec{a}_{2,n}\zeta^2 + \dots$  and  $\phi_{f^n(x)}(\zeta) = f^n(x) + \vec{b}_{1,n}\zeta + \vec{b}_{2,n}\zeta^2 + \dots$ . Then we have  $\psi_{f^n(x)}(\lambda_n \zeta)$  is equal to  $\phi_{f^n(x)}(c\mu_n \zeta)$  up to order  $k$  so by comparing coefficients we get  $\vec{a}_{j,n}\lambda_n^j = \vec{b}_{j,n}c^j\mu_n^j$  for  $j = 1, \dots, k$ . By normality  $|b_{j,n}|$  is bounded by a constant depending on  $j$  but not  $n$  so  $|\vec{a}_{j,n}| \leq C_j \kappa^{-2n} |c^j|$ . We conclude that for  $j = 1, \dots, k$ ,  $|\vec{a}_{j,n}| \rightarrow 0$  as  $n \rightarrow \infty$ . In particular if  $x'$  is in the forward limit set of  $x$  then taking  $\phi_{x'}$  to be a convergent subsequence of  $\phi_{f^n(x)}$  the coefficients of  $\phi_{x'}(\zeta)$  vanish for  $j = 1, \dots, k$ .

If there is a map  $\varphi_x$  in  $\Phi_x^u$  which vanishes to order  $\ell$  then  $\varphi_x(\zeta)$  can be written as the composition of a map  $\xi_x$  and a map  $\alpha$  of order  $\ell$ . Arguing as above we see that the limits  $\varphi_{f^n(x)}\alpha$  vanish to order at least  $(k+1)\ell$ .  $\square$

**Corollary 4.3.** *For  $f$  quasi-hyperbolic there is a bound on the order of tangency between stable and unstable manifolds.*

*Proof.* In [BS8] it is shown that the set  $J_{m^s, m^u}^*$  is empty for  $m^s$  and  $m^u$  large.  $\square$

**Theorem 4.4.** *If  $f$  is quasi-hyperbolic, then the condition that  $J^+$  is laminated in a neighborhood of  $J^*$  is equivalent to the condition that  $J^-$  is laminated in a neighborhood of  $J^*$ ; and either condition is equivalent to hyperbolicity on  $J^*$ .*

*Proof.* If  $f$  is quasi-hyperbolic but not hyperbolic, then by Theorem 2 there is a tangency of order  $k \geq 1$  so by Proposition 4.2 both  $J_{1, k}^*$  and  $J_{k, 1}^*$  are non-empty so neither  $J^+$  nor  $J^-$  are laminated in a neighborhood of  $J^*$ .  $\square$

In [RT] an example is given of a Hénon diffeomorphism for which  $J^+$  is laminated. On the other hand, this example has a semi-parabolic fixed point, which implies that it is not quasi-expanding, and thus we conclude from the results of [BS8] that  $J^-$  is not laminated.

## References

- [BLS] E. Bedford, M. Lyubich and J. Smillie, Polynomial diffeomorphisms of  $\mathbf{C}^2$ . IV: The measure of maximal entropy and laminar currents. *Invent. Math.* 112 (1993), no. 1, 77–125.
- [BS1] E. Bedford and J. Smillie, Polynomial diffeomorphisms of  $\mathbf{C}^2$ : currents, equilibrium measure and hyperbolicity. *Invent. Math.* 103 (1991), no. 1, 69–99.
- [BS3] E. Bedford and J. Smillie, Polynomial diffeomorphisms of  $\mathbf{C}^2$ . III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.* 294 (1992), no. 3, 395–420.
- [BS8] E. Bedford and J. Smillie, Polynomial diffeomorphisms of  $\mathbf{C}^2$ . VIII: Quasi-expansion. *American J. of Math.*, 124, 221–271, (2002).
- [BSr] E. Bedford and J. Smillie, Real polynomial diffeomorphisms with maximal entropy: Tangencies. *Ann. of Math.* (2) 160 (2004), no. 1, 1–26.
- [BD] P. Berger and R. Dujardin, On stability and hyperbolicity for polynomial automorphisms of  $\mathbf{C}^2$ , arXiv:1409.4449
- [CJY] L. Carleson, P. Jones, J.-C. Yoccoz, Julia and John. *Bol. Soc. Brasil. Mat.* (N.S.) 25 (1994), no. 1, 1–30.
- [DL] R. Dujardin and M. Lyubich, Stability and bifurcations for dissipative polynomial automorphisms of  $\mathbf{C}^2$ . *Invent. Math.* 200 (2015), no. 2, 439–511.
- [F] J. E. Fornæss, The Julia set of Hénon maps. *Math. Ann.* 334, 457–464, (2006).
- [FS] J. E. Fornæss and N. Sibony, Complex Hénon mappings in  $\mathbf{C}^2$  and Fatou-Bieberbach domains. *Duke Math. J.* 65 (1992), no. 2, 345–380.
- [FM] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems* 9 (1989), no. 1, 67–99.
- [H] J. Hubbard, The Hénon mapping in the complex domain. *Chaotic dynamics and fractals* (Atlanta, Ga., 1985), 101–111, *Notes Rep. Math. Sci. Engrg.*, 2, Academic Press, Orlando, FL, 1986.
- [LP] M. Lyubich and H. Peters, Classification of invariant Fatou components for dissipative Hénon maps, *Geom. Funct. Anal.* Vol. 24 (2014) 887–915.
- [RT] R. Radu and R. Tanase, A structure theorem for semi-parabolic Hénon maps, arXiv:1411.3824

Eric Bedford  
Stony Brook University  
Stony Brook, NY 11794  
ebedford@math.stonybrook.edu

John Smillie  
Mathematics Institute  
Zeeman Building  
University of Warwick  
Coventry CV4 7AL  
J.Smillie@warwick.ac.uk